Suggested Solutions to:
Regular Exam, Fall 2017
Contract Theory
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Question 1: Adverse selection and optimal procurement with specific functional forms

(a)

Explain in words what each one of the four constraints says and why it must be satisfied. Also explain the nature of the trade-off that the principal faces.

IR-bad: The IR-bad constraint says that the less able type of agent (with $\theta = \overline{\theta}$) must, at least weakly, prefer the contract aimed at her to the outside option. Choosing the contract yields the utility $\overline{t} - C\left(\overline{q}, \overline{\theta}\right)$ and the outside option yields the utility zero. If this condition was violated, the less able type of agent would not choose the contract that P wants her to choose, because the outside option yields a higher utility.

IR-good: The interpretation of the IR-good constraint is analogous to the one for IR-bad, but concerns the relatively able type (with $\theta = \underline{\theta}$).

IC-bad: The IC-bad constraint says that the less able type of agent must, at least weakly, prefer the contract aimed at her to the contract aimed at the relatively able agent. This condition must be satisfied for the less able agent to choose the contract P wants her to choose. P must ensure that this condition is satisfied because P cannot observe the agent's type directly and therefore is unable to instruct the agent to pick one of the two contracts: each agent type must have an incentive to voluntarily choose the one aimed at her.

IC-good: The interpretation of the IC-good constraint is analogous to the one for IC-bad, but concerns the relatively able type.

The trade-off The trade-off is between allocative efficiency and rents (see lecture slides and the textbook for details).

(b)

The first best optimal quantities are defined by $S'\left(\underline{q}^{FB}\right) = C_q\left(\underline{q}^{FB}, \underline{\theta}\right)$ and $S'\left(\overline{q}^{FB}\right) = C_q\left(\overline{q}^{FB}, \overline{\theta}\right)$, respectively. Assume that the constraints (IR-good) and (IC-bad) are lax at the second-best optimum (so that they can be disregarded). Show that, at the second-best optimum, the good type's quantity is *not* distorted relative to the first best ($\underline{q}^{SB} = \underline{q}^{FB}$) and that the bad type's quantity is distorted downwards ($\overline{q}^{SB} < \overline{q}^{FB}$).

The solutions shown below are valid for the case with general cost and surplus functions. It is straightforward, if one so wishes, to plug in the quadratic and linear functions that are specified in the question. (The solutions below indeed makes use of the quadratic cost function in the very end.)

We are allowed to assume that (IR-good) and (IC-bad) are lax at the optimum. Given that, the problem can be written as: Choose $\left(\underline{t}, \underline{q}, \overline{t}, \overline{q}\right)$ so as to maximize

$$\nu\left[S\left(\underline{q}\right) - \underline{t}\right] + (1-\nu)\left[S\left(\overline{q}\right) - \overline{t}\right],$$

subject to the following two constraints:

$$\overline{t} - C\left(\overline{q}, \overline{\theta}\right) \geq 0, \qquad \text{(IR-bad)}$$

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq \overline{t} - C\left(\overline{q}, \underline{\theta}\right). \qquad \text{(IC-good)}$$

Claim: At the optimum of the problem above, both constraints must bind.
Proof of claim:

- Suppose, per contra, that we have an optimum and that IR-bad is lax. Then we can lower $\overline{t}$, while still satisfying both constraints (IC-good will actually be relaxed), thereby increasing the value of the objective function (for this is decreasing in $\overline{t}$). But that is impossible, since we started at an optimum. Hence IR-bad must bind at an optimum.

- Suppose, per contra, that we have an optimum and that IC-good is lax. Then we can lower $\underline{t}$, while still satisfying both constraints (IR-bad will not be affected), thereby increasing the value of the objective function (for this is decreasing in $\underline{t}$). But that is impossible, since we started at an optimum. Hence IC-good must bind at an optimum.

Given that both constraints bind, we can replace the inequalities with equalities and then solve for $\underline{t}$ and $\overline{t}$. Doing this we get:

$$\overline{t} - C\left(\overline{q}, \overline{\theta}\right) = 0 \Rightarrow \overline{t} = C\left(\overline{q}, \overline{\theta}\right)$$

and

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) = \overline{t} - C\left(\overline{q}, \underline{\theta}\right) \Rightarrow \underline{t} = C\left(\underline{q}, \underline{\theta}\right) + \overline{t} - C\left(\overline{q}, \underline{\theta}\right) = C\left(\underline{q}, \underline{\theta}\right) + C\left(\overline{q}, \overline{\theta}\right) - C\left(\overline{q}, \underline{\theta}\right).$$

Plugging these values of $\overline{t}$ and $\underline{t}$ into P's objective function yields

$$\begin{aligned} V &= \nu\left[S\left(\underline{q}\right) - \underline{t}\right] + (1-\nu)\left[S\left(\overline{q}\right) - \overline{t}\right] \\ &= \nu\left[S\left(\underline{q}\right) - C\left(\underline{q}, \underline{\theta}\right) - C\left(\overline{q}, \overline{\theta}\right) + C\left(\overline{q}, \underline{\theta}\right)\right] + (1-\nu)\left[S\left(\overline{q}\right) - C\left(\overline{q}, \overline{\theta}\right)\right] \end{aligned}$$

P's problem is now to maximize the objective V above with respect to only two choice variables, $\underline{q}$ and $\overline{q}$. The first-order condition with respect to $\underline{q}$ is:

$$\frac{\partial V}{\partial \underline{q}} = \nu\left[S'\left(\underline{q}\right) - C_q\left(\underline{q}, \underline{\theta}\right)\right] = 0 \Rightarrow S'\left(\underline{q}^{SB}\right) = C_q\left(\underline{q}^{SB}, \underline{\theta}\right).$$

- This means that $\underline{q}^{SB} = \underline{q}^{FB}$, as we were asked to show.

The first-order condition with respect to $\overline{q}$ is:

$$\frac{\partial V}{\partial \overline{q}} = \nu \left[-C_q\left(\overline{q},\overline{\theta}\right) + C_q\left(\overline{q},\underline{\theta}\right)\right] + (1-\nu)\left[S'\left(\overline{q}\right) - C_q\left(\overline{q},\overline{\theta}\right)\right] = 0$$

or

$$(1-\nu)\, S'\left(\overline{q}\right) = (1-\nu)\, C_q\left(\overline{q},\overline{\theta}\right) + \nu\left[C_q\left(\overline{q},\overline{\theta}\right) - C_q\left(\overline{q},\underline{\theta}\right)\right]$$

or

$$S'\left(\overline{q}^{SB}\right) = C_q\left(\overline{q}^{SB},\underline{\theta}\right) + \frac{\nu}{1-\nu}\left[C_q\left(\overline{q}^{SB},\overline{\theta}\right) - C_q\left(\overline{q}^{SB},\underline{\theta}\right)\right].$$

- From the last equality we see that $\overline{q}^{SB} < \overline{q}^{FB}$ if and only if the last term on the right-hand side is strictly positive. We can write:

$$\frac{\nu}{1-\nu}\left[C_q\left(\overline{q}^{SB},\overline{\theta}\right) - C_q\left(\overline{q}^{SB},\underline{\theta}\right)\right] > 0 \Leftrightarrow C_q\left(\overline{q}^{SB},\overline{\theta}\right) - C_q\left(\overline{q}^{SB},\underline{\theta}\right) > 0 \Leftrightarrow$$

$$\int_{\underline{\theta}}^{\overline{\theta}} C_{q\theta}\left(\overline{q}^{SB},\theta\right) d\theta = \overline{\theta} - \underline{\theta} > 0,$$

  which always holds due to the assumptions that $\overline{\theta} > \underline{\theta}$. This means that we indeed have $\overline{q}^{SB} < \overline{q}^{FB}$, as we were asked to show.

Question 2: Moral hazard in a regulatory problem

(a) Explain in words what behavior outcome 1 and outcome 2, respectively, involves (i.e., explain what the regulator's induced actions x and y are, for each of the two outcomes).

Outcome 1 means that $x = I$ and $y = \theta$. That is, the regulator investigates (thus learns θ) and then awards the monopoly to the firm that yields a positive surplus (whichever that is).

Outcome 2 means that $x = N$ and $y = B$. That is, the regulator does not investigate (and thus knows only the prior distribution of θ) and then always awards the monopoly to firm B.

(b) Solve formally the problem of inducing outcome 1 (i.e., derive the optimal values of t_0, t_A, and t_B, given that outcome 1 should be achieved). Graphical arguments are allowed.

First restate the constraints:

$$t_A \geq (1-\gamma)\, t_B + \gamma\,(t_0 - L)\,, \tag{1}$$

$$t_B \geq t_A, \tag{2}$$

$$\pi t_A + (1-\pi)\, t_B - \psi \geq t_A, \tag{3}$$

$$\pi t_A + (1-\pi)\, t_B - \psi \geq (1-\gamma\pi)\, t_B + \gamma\pi\,(t_0 - L)\,. \tag{4}$$

$$\pi t_A + (1-\pi)\, t_B - \psi \geq 0. \tag{5}$$

In addition, the three limited liability constraints must hold: $t_0 \geq 0$, $t_A \geq 0$, and $t_B \geq 0$. Note that (3) can equivalently be written as $(1-\pi)\,(t_B - t_A) \geq \varphi$, which implies (2); it also, together with $t_A \geq 0$, implies $t_B \geq 0$. Similarly, (4) can equivalently be written as $\pi \left[t_A - (1-\gamma)\, t_B - \gamma \left(t_0 - L\right)\right] \geq \varphi$, which implies (1). Moreover, (3) and $t_A \geq 0$ jointly imply (5). Thus, we can safely ignore the constraints (1), (2), (5), and $t_B \geq 0$.

The government's problem now amounts to maximizing the objective $V = S - \pi t_A - (1-\pi)\, t_B$ w.r.t. t_0, t_A, and t_B, subject to (3), (4) and the limited liability constraints $t_0 \geq 0$ and $t_A \geq 0$.

Claim: At the optimum, the limited liability constraint associated with t_0 is binding: $t_0 = 0$.

Proof of claim: Suppose not—so that, at a solution to the problem, $t_0 > 0$. The variable t_0 does not appear in the objective, and among the constraints (other than $t_0 \geq 0$) it is only (4) that depends on t_0. This constraint is relaxed if we lower t_0. Thus, if $t_0 > 0$ at a solution to the problem, (4) must be lax at that optimum. But if (4) is lax, we must have $t_A = 0$ (otherwise we could profitably lower t_A without violating (4) or any other constraint). Setting $t_A = 0$ in (3) and (4) yields

$$t_B \geq \frac{\varphi}{1-\pi}, \tag{6}$$

$$t_B \leq \frac{\gamma \left(L - t_0\right)}{1-\gamma} + \frac{\varphi}{\pi \left(1-\gamma\right.}. \tag{7}$$

These constraints hold simultaneously for some $t_0 \geq 0$ if, and only if, they hold simultaneously for $t_0 = 0$. But, evaluated at $t_0 = 0$, they cannot hold simultaneously, given the assumptions on the parameters made in the model description. To see this, note that

$$\frac{\varphi}{1-\pi} \leq \frac{\gamma L}{1-\gamma} + \frac{\varphi}{\pi \left(1-\gamma\right.} \Leftrightarrow \frac{\gamma \pi (1-\pi)}{1-\gamma\pi} \geq \frac{\varphi}{L}.$$

Under the assumptions that $L < \varphi$, $\gamma \in (0,1)$, and $\pi \in (0, \frac{1}{2})$, the right-hand side is larger than one while the left-hand-side is smaller than one. It follows that if $t_A = 0$, then at least one of the constraints is violated. Hence $t_A > 0$, which also means that $t_0 = 0$. □

We now have two remaining choice variables, t_A and t_B. Given $t_0 = 0$, the constraints (3) and (4) can be rewritten as

$$t_A \leq t_B - \frac{\varphi}{1-\pi}, \tag{8}$$

$$t_A \geq (1-\gamma)\, t_B + \frac{\varphi}{\pi} - \gamma L. \tag{9}$$

The two constraints (8) and (9) are both upward-sloping in the (t_B, t_A)-space and the latter is flatter. Moreover, under the assumption $\varphi > L$, their intersection lies inside the positive quadrant, where the limited liability constraint $t_A \geq 0$ is satisfied. The intersection is given by

$$(t_A, t_B) = \left(\frac{\varphi \left(1-\gamma\pi\right)}{\gamma\pi \left(1-\pi\right)} - L, \frac{\varphi}{\gamma\pi \left(1-\pi\right)} - L \right) \tag{10}$$

and it constitutes the south-west corner of the feasible set. The principal's indifference curve is downward-sloping, which means that the solution to the problem is given by (10) and $t_0 = 0$. (To illustrate this, one can draw a figure.)

(c) Solve formally the problem of inducing outcome 2 (i.e., derive the optimal values of t_0, t_A, and t_B, given that outcome 2 should be achieved). Graphical arguments are allowed.

First restate the constraints:

$$(1-\gamma\pi)\, t_B + \gamma\pi \left(t_0 - L\right) \geq t_A, \tag{11}$$

$$(1 - \gamma\pi)\, t_B + \gamma\pi\, (t_0 - L) \geq \pi t_A + (1 - \pi)\, t_B - \psi, \tag{12}$$

$$(1 - \gamma\pi)\, t_B + \gamma\pi\, (t_0 - L) \geq 0, \tag{13}$$

In addition, the three limited liability constraints must hold: $t_0 \geq 0$, $t_A \geq 0$, and $t_B \geq 0$. The objective does not contain t_A, whereas all the constraints except for $t_A \geq 0$ are either relaxed or unaffected by a decrease of this variable. Therefore, at the optimum, $t_A = 0$. Using $t_A = 0$, constraint (12) can equivalently be written as $\pi\, [(1 - \gamma)\, t_B + \gamma t_0] + \varphi - \gamma\pi L \geq 0$. But this inequality must hold, due to the limited liability constraints and the assumption that $\varphi > L$. Hence, we can safely ignore (12). We can also disregard (11), as it coincides with (13).

The government's problem now amounts to maximizing the objective $V = (1 - \pi)\, S - (1 - \gamma\pi)\, t_B - \gamma\pi t_0$ w.r.t. t_0 and t_B, subject to (13), and $t_0 \geq 0$ and $t_B \geq 0$. The constraint (13) can be rewritten as

$$t_0 \geq L - \frac{1 - \gamma\pi}{\gamma\pi} t_B.$$

This equation has the same slope as the principal's indifference curve, which means that the set of optimal contracts is characterized by $t_A = 0$, $t_B \geq 0$, $t_0 \geq 0$, and $(1 - \gamma\pi)\, t_B + \gamma\pi t_0 = \gamma\pi L$. (To illustrate this, one can draw a figure.)

(d) Explain (in words) the economic reasons for the result described immediately above. Why is there a lack of monotonicity? Why does Firm B prefer either a low or a high value of L to an intermediate value?

The parameter L is a personal penalty that the regulator must incur if awarding the monopoly to B when the state is in favor of A. This means that the cost L is effectively an exogenous incentive that makes it risky for the regulator to award the monopoly to B, unless she knows for sure that the state is $\theta = B$. In particular, if the regulator is never paid anything from the government (so if $t_0 = t_A = t_B = 0$), she will not investigate ($x = N$) and then choose $y = A$. Call this Outcome 3. Thus, the government can induce Outcome 3 for free, thanks to the positive L.

The government may, however, prefer to incentivize the regulator either to (i) not investigate and then choose $y = B$ [Outcome 2], or (ii) investigate and then choose $y = \theta$ [Outcome 1].

- To induce Outcome 2 is inexpensive for the government if L is small, for then t_B doesn't have to be very large in order to compensate the regulator for the exogenous cost associated with L. Thus, if L is small we may expect Outcome 2 to be induced, which obviously is very good for firm B.

- To induce Outcome 1 should be relatively inexpensive for the government if L is large, since some of the incentives to choose $y = A$ when $\theta = A$ are taken care of by the presence of L—there is no need to make t_A very large. Therefore, for large values of L, we may expect Outcome 1 to be optimally induced by the government, which is also good for B.

- The above reasoning suggest a reason for why firm B gets desirable outcomes (Outcomes 1 or 2) if L is either small or large, but an undesirable outcome (Outcome 3) for intermediate values of L. It is not obvious though that, as we gradually increase L, Outcome 3 becomes optimal before Outcome 1 does. However, this is a possibility and it is consistent with our reasoning.[1]

[1] By solving the full model one can show that for certain parameter values Outcome 3 indeed becomes optimal before Outcome 1 does, although showing this is of course not required.